Eigenvectors in the superintegrable model II: ground-state sector

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Abstract

In 1993, Baxter gave 2^{m_Q} eigenvalues of the transfer matrix of the N-state superintegrable chiral Potts model with the spin-translation quantum number Q, where $m_Q = \lfloor (NL - L - Q)/N \rfloor$. In our previous paper we studied the $Q = 0$ ground-state sector, when the size L of the transfer matrix is chosen to be a multiple of N. It was shown that the corresponding τ_2 matrix has a degenerate eigenspace generated by the generators of $r = m_0$ simple $\mathfrak{sl}_2$ algebras. These results enable us to express the transfer matrix in the subspace in terms of these generators $\mathbf{E}_m^\pm$ and $\mathbf{H}_m$ for $m = 1, \ldots, r$. Moreover, the corresponding 2^r eigenvectors of the transfer matrix are expressed in terms of rotated eigenvectors of $\mathbf{H}_m$.

In our previous paper [1], we have shown that to calculate the correlation functions of the chiral Potts model, one only needs to study the eigenvectors of the superintegrable model. Since the transfer matrix and the matrix $\tau_2(t_q)$ have the same eigenvectors, we have examined the eigenspace of $\tau_2(t_q)$ which has degeneracy 2^r, where $r = L(N-1)/N$ with L—the length of the transfer matrix—chosen to be a multiple of N. This space is a highest-weight representation of an $\mathcal{L}(\mathfrak{sl}_2)$ loop algebra which turns out to be a direct sum of r copies of the simple algebra $\mathfrak{sl}_2$ whose generators are $\mathbf{E}_m^\pm$ and $\mathbf{H}_m$ for $m = 1, \ldots, r$.

In 1989, Baxter in his paper on the 'Superintegrable Chiral Potts Model' [2] suggested that the transfer matrix restricted to a certain subspace may be written as

$$\mathcal{T}_Q = C_Q \rho^L \prod_{j=1}^r \left\{ GI + (1 - k' \cos \theta_j)\sigma_j^z - k' \sin \theta_j \sigma_j^x \right\} \tag{1}$$

(see equation (2.16) of [2]). Our attempt is to do exactly that. However, as the modulus k' in that paper [2] is different from the usual k' of other papers, we shall consider the transfer matrix of his later papers [3, 4].

From equation (6.2) of [3][3] the transfer matrices of the superintegrable model can be written as

$$T_q = N^{\frac{1}{2}L} \frac{(x_q - y_p)^L}{\left(x_q^N - y_p^N\right)^L} \mathcal{T}(x_q, y_q), \qquad \hat{T}_q = N^{\frac{1}{2}L} \frac{(x_q - x_p)^L}{\left(x_q^N - x_p^N\right)^L} \hat{\mathcal{T}}(x_q, y_q). \tag{2}$$

Baxter showed that the eigenvalues of $\mathcal{T}(x_q, y_q)$ for spin shift quantum number $Q = 0$, or equivalently $\mathcal{T}_0(x_q, y_q)$ corresponding to (I.10)[4], depend on $\lambda_q = \mu_q^N$ only and are denoted by $\mathcal{G}(\lambda_q)$. The corresponding eigenvalue $\hat{\mathcal{G}}(\lambda_q)$ of $\hat{\mathcal{T}}_0(x_q, y_q)$ can be made equal to $\mathcal{G}(\lambda_q)$ by simple rescaling of the eigenvectors $\mathbf{x}$ and $\mathbf{y}$, so that

$$\mathcal{T}_0(x_q, y_q)\mathbf{x} = \mathcal{G}(\lambda_q)\mathbf{y}, \qquad \hat{\mathcal{T}}_0(x_q, y_q)\mathbf{y} = \mathcal{G}(\lambda_q)\mathbf{x}, \qquad \hat{\mathcal{T}}_0(y_q, x_q)\mathbf{y} = \mathcal{G}\big(\lambda_q^{-1}\big)\mathbf{x}. \tag{3}$$

Thus, (I.9) and (6.24) of [3] become[5]

$$\mathcal{G}(\lambda_q)\mathcal{G}\big(\lambda_q^{-1}\big) = t_p^{rN} N \mathcal{P}(t_q/t_p) = t_p^{rN} N \prod_{j=1}^{r} \left[(t_q/t_p)^N - z_j\right], \tag{4}$$

where z_j and z_j^{-1} are the roots of the Drinfeld polynomial $P(z) = P(t^N) = \mathcal{P}(t)$, see (I.11) with $Q = 0$ and (I.41). From (I.2) we have

$$k^2 t_q^N = 1 + k'^2 - k'\big(\lambda_q + \lambda_q^{-1}\big). \tag{5}$$

Substituting this equation into (4), and letting

$$2\cosh 2\theta_j = k' + k'^{-1} - k^2 t_p^N z_j / k', \tag{6}$$

we find

$$\mathcal{G}(\lambda_q)\mathcal{G}\big(\lambda_q^{-1}\big) = N \left(\frac{k'}{k^2}\right)^r \prod_{j=1}^{r} e^{\mp 2\theta_j}\big(e^{\pm 2\theta_j} - \lambda_q^{-1}\big)(e^{\pm 2\theta_j} - \lambda_q). \tag{7}$$

Therefore, as $\mathcal{G}(\lambda_q)$ is a polynomial in λ_q^{-1} according to section 6 of [3], the 2^r implied eigenvalues of the transfer matrix $\mathcal{T}_0(x_q, y_q)$ in (I.10) are

$$\mathcal{G}(\lambda_q) = \rho^r \prod_{j=1}^{r} \left[\cosh\theta_j\big(1 - \lambda_q^{-1}\big) \pm \sinh\theta_j\big(1 + \lambda_q^{-1}\big)\right] = \prod_{j=1}^{r}(A_j \pm B_j), \tag{8}$$

$$A_j = \rho \cosh\theta_j\big(1 - \lambda_q^{-1}\big), \qquad B_j = \rho \sinh\theta_j\big(1 + \lambda_q^{-1}\big), \qquad \rho^r = N^{\frac{1}{2}}(k'/k^2)^{\frac{1}{2}r}, \tag{9}$$

cf (20) of [4]. These are the results of Baxter, written in different forms.

We shall now attempt to express the transfer matrix $\mathcal{T}_0(x_q, y_q)$ of (I.10) for $Q = 0$ in terms of the generators $\mathbf{E}_m^{\pm}$ and $\mathbf{H}_m$, defined for $m = 1, \ldots, r$ in [1], and also obtain the corresponding eigenvectors.

Similar to what Davies [5] did, we define the polynomials

$$f_j(z) = \prod_{\ell \neq j} \frac{z - z_\ell}{z_j - z_\ell} = \sum_{n=0}^{r-1} \beta_{j,n} z^n, \qquad f_j(z_k) = \delta_{j,k}, \tag{10}$$

---

[3] This simplifies under the periodic boundary condition, i.e. $r = m_P = 0$ in [3] with r being Baxter's skewness parameter here, not to be confused with $r = L(N - 1)/N$ in this paper.
[4] All equations in [1] are denoted here by prefacing I to its equation number.
[5] Equation (I.11) with $Q = 0$, $t \equiv t_q/t_p$ and (6.25) of [3] with $P_a = P_b = 0$, $F(t_q) \equiv 1$ differ by a factor N.

where $\beta_{j,n}$ are the elements of the inverse of the Vandermonde matrix [6], such that

$$\sum_{n=0}^{r-1} \beta_{j,n} z_k^n = \delta_{j,k}, \qquad \sum_{k=1}^{r} z_k^n \beta_{k,m} = \delta_{n,m} \qquad \text{for} \quad 0 \leqslant n \leqslant r-1. \tag{11}$$

Consequently, (I.45)[6] rewritten here as

$$\mathbf{x}_n^{\mp} = \pm \sum_{m=1}^{r} z_m^{-n} \mathbf{E}_m^{\pm} = \pm \sum_{m=1}^{r} z_{m^*}^n \mathbf{E}_m^{\pm}, \qquad \mathbf{h}_n = \sum_{m=1}^{r} z_m^{-n} \mathbf{H}_m = \sum_{m=1}^{r} z_{m^*}^n \mathbf{H}_m, \tag{12}$$

where $n \in \mathbb{Z}$ and m^* is the subscript for which $z_{m^*} = 1/z_m$, can be inverted as

$$\mathbf{E}_m^{\pm} = \pm \sum_{n=0}^{r-1} \beta_{m^*,n} z_m^{\ell} \mathbf{x}_{n+\ell}^{\mp}, \qquad \mathbf{H}_m = \sum_{n=0}^{r-1} \beta_{m^*,n} z_m^{\ell} \mathbf{h}_{n+\ell}, \tag{13}$$

valid for $m = 1, \ldots, r$ and $\ell \in \mathbb{Z}$.

In appendix A, we shall show that

$$\mathbf{H}_m |\Omega\rangle = -|\Omega\rangle, \qquad \left(\mathbf{E}_m^+\right)^2 |\Omega\rangle = 0, \tag{14}$$

with $|\Omega\rangle$ the 'ferromagnetic' ground state of [1], whereas in appendix B we shall give some further results for $\mathbf{E}_j^+ \mathbf{E}_m^+ |\Omega\rangle$.

Deguchi and Nishino have studied a similar loop algebra [7–10]. They have shown that there exists a transformation to map the generators of the two loop algebras [10]. The proofs of the identities for the related operators are also given in [9]. (It can be verified that our $\mathbf{E}_j^{\pm}$ is proportional to some $\rho_j^{\mp}(\mathbf{a}; s)$ of [9], noting that Deguchi essentially uses Prony's method [6] to invert the Vandermonde matrix, but leaves out the denominators in [6].)

From the commutation relations,

$$\left[\mathbf{E}_m^+, \mathbf{E}_n^-\right] = \delta_{m,n} \mathbf{H}_m, \qquad \left[\mathbf{H}_m, \mathbf{E}_n^{\pm}\right] = \pm 2 \delta_{m,n} \mathbf{E}_m^{\pm}, \qquad \left[\mathbf{E}_m^+, \mathbf{E}_n^+\right] = [\mathbf{E}_m^-, \mathbf{E}_n^-] = 0, \tag{15}$$

cf (I.46), we find

$$\mathbf{H}_m \mathbf{E}_n^+ |\Omega\rangle = -(-1)^{\delta_{m,n}} \mathbf{E}_n^+ |\Omega\rangle. \tag{16}$$

Thus by operating with various products given by the subsets of $\mathbf{E}_1^+, \mathbf{E}_2^+, \ldots, \mathbf{E}_r^+$ on $|\Omega\rangle$, we obtain the eigenvectors of $\mathbf{H}_m$, with eigenvalues ξ_m and $\xi_m = \pm 1$ depending on whether m is in the subset or not. Following the example of Onsager [11], the 2^r eigenvectors are denoted by the eigenvalues ξ_m, namely,

$$\Psi(\xi_1, \xi_2, \ldots, \xi_r) = \prod_{m \in J_n} \mathbf{E}_m^+ |\Omega\rangle, \qquad \xi_j = \pm 1, \tag{17}$$

where $J_n = (j_1, \ldots, j_n)$ is any subset of $(1, 2, \ldots, r)$ for $n = 0, \ldots, r$, with $\xi_j = 1$ for $j \in J_n$, and $\xi_j = -1$ for $j \notin J_n$. Equivalently,

$$\mathbf{H}_j \Psi(\xi_1, \xi_2, \ldots, \xi_r) = \xi_j \Psi(\xi_1, \xi_2, \ldots, \xi_r). \tag{18}$$

Particularly, for $n = 0$ and $n = r$, we have

$$|\Omega\rangle = \Psi(-1, -1, \ldots, -1), \qquad |\bar{\Omega}\rangle = \Psi(1, 1, \ldots, 1). \tag{19}$$

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[6] We have replaced $\mathbf{E}_m^{\mp} \to \pm \mathbf{E}_m^{\pm}$ in (I.45), so that we obtain the more usual (15) rather than (I.46). Also, z_m and $1/z_m$ are both zeros of $P(z)$ when $Q = 0$, because of the symmetry of the coefficients of the polynomial $P(z)$ in (I.41), so that (12) and (I.45) are equivalent. The changes made seem to be appropriate for the $Q \neq 0$ case.

Because $\xi_j = \pm 1$, it is easy to see that $\mathbf{H}_j^2 = \mathbf{1}$ in this restricted eigenspace. Since $\mathbf{E}_j^-|\Omega\rangle = 0$ and $\mathbf{H}_j|\Omega\rangle = -|\Omega\rangle$, we find from (15)

$$
\begin{aligned}
\mathbf{E}_j^- \Psi(\xi_1, \xi_2, \ldots, \xi_r) &= \mathbf{E}_j^- \prod_{m \in J_n} \mathbf{E}_m^+ |\Omega\rangle \\
&= \begin{cases} \Psi(\xi_1, \ldots, \xi_{j-1}, -\xi_j, \xi_{j+1}, \ldots, \xi_r) & \text{if } j \in J_n, \\ 0 & \text{if } j \notin J_n. \end{cases}
\end{aligned}
\tag{20}
$$

Likewise, using $\left(\mathbf{E}_j^+\right)^2 |\Omega\rangle = 0$ we obtain

$$
\begin{aligned}
\mathbf{E}_j^+ \Psi(\xi_1, \xi_2, \ldots, \xi_r) &= \mathbf{E}_j^+ \prod_{m \in J_n} \mathbf{E}_m^+ |\Omega\rangle \\
&= \begin{cases} 0 & \text{if } j \in J_n, \\ \Psi(\xi_1, \ldots, \xi_{j-1}, -\xi_j, \xi_{j+1}, \ldots, \xi_r) & \text{if } j \notin J_n. \end{cases}
\end{aligned}
\tag{21}
$$

Consequently, we find

$$
\left(\mathbf{E}_j^+ + \mathbf{E}_j^-\right)\Psi(\xi_1, \xi_2, \ldots, \xi_r) = \Psi(\xi_1, \ldots, \xi_{j-1}, -\xi_j, \xi_{j+1}, \ldots, \xi_r),
\tag{22}
$$

and

$$
\left(\mathbf{E}_j^+ + \mathbf{E}_j^-\right)^2 \Psi(\xi_1, \xi_2, \ldots, \xi_r) = \Psi(\xi_1, \ldots, \xi_j, \ldots, \xi_r).
\tag{23}
$$

From the commutation relation (15), we can show

$$
\left[\mathbf{H}_j, \left(\mathbf{E}_j^+ + \mathbf{E}_j^-\right)\right] = 2\left(\mathbf{E}_j^+ - \mathbf{E}_j^-\right), \qquad \left[\mathbf{H}_j, \left(\mathbf{E}_j^+ - \mathbf{E}_j^-\right)\right] = 2\left(\mathbf{E}_j^+ + \mathbf{E}_j^-\right),
\tag{24}
$$

$$
\left[\left(\mathbf{E}_j^+ - \mathbf{E}_j^-\right), \left(\mathbf{E}_j^+ + \mathbf{E}_j^-\right)\right] = 2\mathbf{H}_j, \qquad \left(\mathbf{E}_j^+ - \mathbf{E}_j^-\right)^2 = -1.
\tag{25}
$$

Now, similar to what Onsager did for the Ising model in [11], we express the transfer matrix $\mathcal{T}_0(x_q, y_q)$ in (I.10) within the eigenspace of the eigenvalues given in (8) as

$$
\mathcal{T}_0(x_q, y_q) = \mathcal{S} \prod_{j=1}^r (A_j - \mathbf{H}_j B_j)\mathcal{R}^{-1},
\tag{26}
$$

$$
\hat{\mathcal{T}}_0(x_q, y_q) = \mathcal{R} \prod_{j=1}^r (A_j - \mathbf{H}_j B_j)\mathcal{S}^{-1}, \qquad \hat{\mathcal{T}}_0(y_q, x_q) = \mathcal{R} \prod_{j=1}^r (\bar{A}_j - \mathbf{H}_j \bar{B}_j)\mathcal{S}^{-1},
\tag{27}
$$

where $\mathcal{S}$ and $\mathcal{R}$ are some complex rotation operators, which shall be determined, and $\bar{A}_j$ and $\bar{B}_j$ in (27) can be obtained from A_j and B_j in (9) by letting $\lambda_q^{-1} \to \lambda_q$.

If we introduce

$$
|\mathcal{X}_i\rangle = \mathcal{R}\Psi(\xi_1, \xi_2, \ldots, \xi_r), \qquad |\mathcal{Y}_i\rangle = \mathcal{S}\Psi(\xi_1, \xi_2, \ldots, \xi_r),
\tag{28}
$$

for $i = 1, \ldots, 2^r$, then we find from (18) and (26), cf also (3), that

$$
\mathcal{T}_0(x_q, y_q)|\mathcal{X}_i\rangle = \mathcal{G}(\lambda_q, \{\xi_j\})|\mathcal{Y}_i\rangle, \qquad \hat{\mathcal{T}}_0(x_q, y_q)|\mathcal{Y}_i\rangle = \mathcal{G}(\lambda_q, \{\xi_j\})|\mathcal{X}_i\rangle,
\tag{29}
$$

where the eigenvalues are

$$
\mathcal{G}(\lambda_q, \{\xi_j\}) = \prod_{j=1}^r (A_j - \xi_j B_j).
\tag{30}
$$

Particularly, the 'largest' eigenvalue is

$$
\mathcal{G}_0(\lambda_q) = \mathcal{G}(\lambda_q, \{-1\}) = \prod_{j=1}^r (A_j + B_j),
\tag{31}
$$

4

with the ground-state eigenvector

$$|\mathcal{X}_1\rangle = \mathcal{R}\Psi(-1, -1, \ldots, -1) = \mathcal{R}|\Omega\rangle. \tag{32}$$

The rotations $\mathcal{R}$ and $\mathcal{S}$ should be chosen such that they are independent of q as the transfer matrices with different q commute. They are determined from the explicit form of the transfer matrices given in (2). We now calculate the matrix elements of the transfer matrices. Using (I.4) and (I.10), we rewrite the transfer matrices in (2) (from configuration $\{\sigma\}$ to $\{\sigma'\}$ in the row above) in terms of the edge variables $\{n_i\}$ and $\{n'_i\}$ with $n_i = \sigma_i - \sigma_{i+1}$ as

$$\langle\{n'_i\}|\mathcal{T}_Q(x_q, y_q)|\{n_i\}\rangle = N^{-\frac{1}{2}L}\sum_{a=0}^{N-1}\omega^{-Qa}\prod_{j=1}^{L}\left[(y_p^N - x_q^N)(y_p - x_q)^{-1}\right.$$
$$\left.\times W_{pq}(a - N_j + N'_j)\overline{W}_{p'q}(a - N_{j+1} + N'_j)\right], \qquad N_j = \sum_{\ell < j} n_\ell = \sigma_1 - \sigma_j, \tag{33}$$

where $a = \sigma_1 - \sigma'_1$, and

$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n\prod_{j=1}^{n}\frac{y_q - x_p\omega^j}{y_p - x_q\omega^j}, \qquad \overline{W}_{p'q}(n) = \left(\frac{\mu_q}{\mu_p}\right)^n\prod_{j=1}^{n}\frac{\omega y_p - x_q\omega^j}{y_q - x_p\omega^j}. \tag{34}$$

Canceling out the common factors in the weights, (33) becomes

$$\langle\{n'_i\}|\mathcal{T}_Q(x_q, y_q)|\{n_i\}\rangle = N^{-\frac{1}{2}L}\sum_{a=0}^{N-1}\omega^{-Qa}$$
$$\times\prod_{j=1}^{L}\left[\frac{(y_p^N - x_q^N)\omega^{a-N_{j+1}+N'_j}}{y_p - x_q\omega^{a-N_{j+1}+N'_j}}\left(\frac{\mu_p}{\mu_q}\right)^{n_j}\prod_{\ell=1}^{n_j}\frac{y_q - x_p\omega^{\ell+a-N_{j+1}+N'_j}}{y_p - x_q\omega^{\ell+a-N_{j+1}+N'_j}}\right]. \tag{35}$$

It is easy to see that for the two ground states,

$$|\Omega\rangle \leftrightarrow n_i \equiv N_i \equiv 0 \qquad \text{or} \qquad |\bar{\Omega}\rangle \leftrightarrow n_i \equiv N - 1, \qquad N_i \equiv (i - 1)(N - 1), \tag{36}$$

the above expression simplifies to

$$\langle\{n_i\}|\mathcal{T}_Q(x_q, y_q)|\Omega\rangle = N^{-\frac{1}{2}L}\sum_{a=0}^{N-1}\omega^{-Qa}\prod_{j=1}^{L}\frac{\omega^{a+N_j}(y_p^N - x_q^N)}{y_p - x_q\omega^{a+N_j}}, \tag{37}$$

and

$$\langle\{n_i\}|\mathcal{T}_Q(x_q, y_q)|\bar{\Omega}\rangle = N^{-\frac{1}{2}L}\sum_{a=0}^{N-1}\omega^{-Qa}\prod_{j=1}^{L}\left(\frac{\mu_p}{\mu_q}\right)^{N-1}\frac{\omega^{a+j+N_j}(y_q^N - x_p^N)}{y_q - x_p\omega^{a+j+N_j}}, \tag{38}$$

where also $n_1 + \cdots + n_L \equiv 0 \pmod{N}$, as is required by the periodic boundary conditions. For (38) we have used $\prod_{\ell=1}^{N}(y - x\omega^\ell) = y^N - x^N$. Particularly, when $n_1 = \cdots = n_L = 0$, we find $N_j = 0$ and (37) becomes

$$\langle\Omega|\mathcal{T}_Q(x_q, y_q)|\Omega\rangle = N^{1-\frac{1}{2}L}y_p^{rN}(x_q/y_p)^Q\mathcal{P}_Q(x_q/y_p), \tag{39}$$

while for $n_1 = \cdots = n_L = N - 1$, we substitute $N_j = (j - 1)(N - 1)$ into (37) to obtain

$$\langle\bar{\Omega}|\mathcal{T}_Q(x_q, y_q)|\Omega\rangle = N^{1-\frac{1}{2}L}\delta_{Q,0}\omega^{-L(L+1)/2}(y_p^N - x_q^N)^r. \tag{40}$$

Similarly we find from (38)

$$\langle\bar{\Omega}|\mathcal{T}_Q(x_q, y_q)|\bar{\Omega}\rangle = N^{1-\frac{1}{2}L}\omega^Q(\mu_p y_q/\mu_q)^{rN}(x_p/y_q)^Q\mathcal{P}_Q(x_p/y_q) \tag{41}$$

and

$$\langle\Omega|\mathcal{T}_Q(x_q, y_q)|\bar{\Omega}\rangle = N^{1-\frac{1}{2}L}\delta_{Q,0}\omega^{L(L+1)/2}\big(y_p^N - x_q^N\big)^r. \tag{42}$$

Since L is a multiple of N, the right-hand sides of (40) and (42) are equal; this means that the signs of $\mathbf{E}_j^+$ and $\mathbf{E}_j^-$ are equal, and that the transfer matrix can only have the following form for the 2^r-dimensional ground-state sector:

$$\mathcal{T}_0(x_q, y_q) = \prod_{j=1}^{r}\big[X_j - \mathbf{H}_j Y_j + \big(\mathbf{E}_j^+ + \mathbf{E}_j^-\big)Z_j\big]. \tag{43}$$

To evaluate X_j, Y_j and Z_j, we need to evaluate $\langle\Omega|\mathbf{E}_m^-$ and $\langle\bar{\Omega}|\mathbf{E}_m^+$. To do so, we invert both (I.42) and

$$\langle\Omega|(\mathbf{x}_0^+)^{(n)}(\mathbf{x}_1^-)^{(n-1)} = \sum_{j=1}^{n}\Lambda_{n-j}\langle\Omega|\mathbf{x}_{j-1}^+, \qquad \langle\bar{\Omega}|(\mathbf{x}_1^-)^{(n)}(\mathbf{x}_0^+)^{(n-1)} = \sum_{j=1}^{n}\Lambda_{n-j}\langle\bar{\Omega}|\mathbf{x}_j^-, \tag{44}$$

which can be derived similarly, as

$$\mathbf{x}_j^-|\Omega\rangle = \sum_{n=1}^{j}S_{j-n}(\mathbf{x}_0^+)^{(n-1)}(\mathbf{x}_1^-)^{(n)}|\Omega\rangle, \qquad \mathbf{x}_{j-1}^+|\bar{\Omega}\rangle = \sum_{n=1}^{j}S_{j-n}(\mathbf{x}_1^-)^{(n-1)}(\mathbf{x}_0^+)^{(n)}|\bar{\Omega}\rangle, \tag{45}$$

$$\langle\Omega|\mathbf{x}_{j-1}^+ = \sum_{n=1}^{j}S_{j-n}\langle\Omega|(\mathbf{x}_0^+)^{(n)}(\mathbf{x}_1^-)^{(n-1)}, \qquad \langle\bar{\Omega}|\mathbf{x}_j^- = \sum_{n=1}^{j}S_{j-n}\langle\bar{\Omega}|(\mathbf{x}_1^-)^{(n)}(\mathbf{x}_0^+)^{(n-1)}, \tag{46}$$

where

$$\sum_{n=0}^{m}\Lambda_{m-n}S_n = \delta_{m,0}, \qquad \text{with} \qquad S_0 = 1, \qquad \Lambda_0 = 1. \tag{47}$$

This is most easily seen by viewing $\boldsymbol{\Lambda}$ as a lower triangular $r + 1$ by $r + 1$ matrix with elements $\Lambda_{i,j} = \Lambda_{i-j}$, zero whenever $j > i$, and a similar form for its inverse. Letting

$$P(z) = \sum_{n=0}^{r}\Lambda_n z^n, \qquad Q(z) = \sum_{n=0}^{r}S_n z^n, \tag{48}$$

we find from (47) that

$$Q(z) = 1/P(z) = \prod_{n=1}^{r}(z - z_n)^{-1} = \sum_{i=1}^{r}(z - z_i)^{-1}\prod_{\ell\neq i}^{r}(z_i - z_\ell)^{-1}. \tag{49}$$

Using (10) with $z = 0$ together with $\prod_{\ell=1}^{r}(-z_\ell) = 1$, we obtain

$$S_n = \sum_{i=1}^{r}z_i^{-n}\beta_{i,0}, \qquad \text{for} \quad 1-r < n < r-1, \tag{50}$$

also using (11), which implies $S_n = 0$ for $1-r < n < 0$. Substituting (46) into (13) with $\ell = 0$, interchanging the order of summation, and using (50), we get

$$\langle\Omega|\mathbf{E}_m^- = -\langle\Omega|\sum_{\ell=1}^{r}(\mathbf{x}_0^+)^{(\ell)}(\mathbf{x}_1^-)^{(\ell-1)}\sum_{i=1}^{r}\beta_{i,0}\sum_{j=0}^{r-1}\beta_{m^*,j}z_i^{\ell-j-1}, \tag{51}$$

where the condition $S_n = 0$ for negative values of n has been used to extend the interval of the j summation. As (I.41) implies the symmetry $P(z) = z^r P(1/z)$, we find that for each root z_i of $P(z)$, $z_{i^*} \equiv 1/z_i$ is also a root. We now use the first equation in (11), i.e.

$$\sum_{j=0}^{r-1} \beta_{m^*,j} z_i^{-j} = \sum_{j=0}^{r-1} \beta_{m^*,j} z_{i^*}^{j} = \delta_{m,i}, \tag{52}$$

to get

$$\langle \Omega | \mathbf{E}_m^- = -\beta_{m,0} \sum_{\ell=1}^{r} z_m^{\ell-1} \langle \Omega | (\mathbf{x}_0^+)^{(\ell)} (\mathbf{x}_1^-)^{(\ell-1)},$$
$$\mathbf{E}_m^- | \bar{\Omega} \rangle = -\beta_{m,0} \sum_{\ell=1}^{r} z_m^{\ell-1} (\mathbf{x}_1^-)^{(\ell-1)} (\mathbf{x}_0^+)^{(\ell)} | \bar{\Omega} \rangle. \tag{53}$$

Similarly, but now using (13) with $\ell = 1$, we can show

$$\mathbf{E}_m^+ | \Omega \rangle = \beta_{m,0} \sum_{\ell=1}^{r} z_m^{\ell} (\mathbf{x}_0^+)^{(\ell-1)} (\mathbf{x}_1^-)^{(\ell)} | \Omega \rangle,$$
$$\langle \bar{\Omega} | \mathbf{E}_m^+ = \beta_{m,0} \sum_{\ell=1}^{r} z_m^{\ell} \langle \bar{\Omega} | (\mathbf{x}_1^-)^{(\ell)} (\mathbf{x}_0^+)^{(\ell-1)}. \tag{54}$$

From (I.20), we find[7]

$$\frac{\mathfrak{e}^n}{[n]!} | n' \rangle = \begin{bmatrix} n' \\ n \end{bmatrix} | n' - n \rangle, \qquad \frac{\mathfrak{f}^n}{[n]!} | n' \rangle = \begin{bmatrix} n' + n \\ n \end{bmatrix} | n' + n \rangle,$$
$$\langle n' | \frac{\mathfrak{e}^n}{[n]!} = \begin{bmatrix} n' + n \\ n \end{bmatrix} \langle n' + n |, \qquad \langle n' | \frac{\mathfrak{f}^n}{[n]!} = \begin{bmatrix} n' \\ n \end{bmatrix} \langle n' - n |. \tag{55}$$

Using (I.35)[8] and the above formulas, we obtain

$$(\mathbf{x}_1^-)^{(\ell)} | \Omega \rangle = \sum_{\substack{\{0 \leqslant n_j \leqslant N-1\} \\ n_1 + \cdots + n_L = \ell N}} \omega^{-\sum_j j n_j} | \{n_j\} \rangle,$$
$$(\mathbf{x}_0^+)^{(\ell)} | \bar{\Omega} \rangle = (-1)^\ell \sum_{\substack{\{0 \leqslant n_j \leqslant N-1\} \\ n_1 + \cdots + n_L = \ell N}} | \{N - 1 - n_j\} \rangle, \tag{56}$$
$$\langle \Omega | (\mathbf{x}_0^+)^{(\ell)} = \sum_{\substack{\{0 \leqslant n_j \leqslant N-1\} \\ n_1 + \cdots + n_L = \ell N}} \omega^{\sum_j j n_j} \langle \{n_j\} |,$$
$$\langle \bar{\Omega} | (\mathbf{x}_1^-)^{(\ell)} = (-1)^\ell \sum_{\substack{\{0 \leqslant n_j \leqslant N-1\} \\ n_1 + \cdots + n_L = \ell N}} \langle \{N - 1 - n_j\} |, \tag{57}$$

where we have twice used

$$\begin{bmatrix} N - 1 - n \\ \nu \end{bmatrix} = (-1)^\nu \omega^{-n\nu - \nu(\nu+1)/2} \begin{bmatrix} \nu + n \\ \nu \end{bmatrix} \tag{58}$$

with $n = 0$. Continuing the same process we find

$$(\mathbf{x}_0^+)^{(\ell)} (\mathbf{x}_1^-)^{(\ell+1)} | \Omega \rangle = \sum_{\substack{\{0 \leqslant n_j \leqslant N-1\} \\ n_1 + \cdots + n_L = N}} \omega^{-\sum_j j n_j} K_{\ell N}(\{n_j\}) | \{n_j\} \rangle,$$

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[7] We use the standard definition $\begin{bmatrix} n' \\ n \end{bmatrix} \equiv \frac{[n']!}{[n]![n'-n]!}$.

[8] There are some misprints in (I.35): The four subscripts j must be replaced by m.

$$\langle\Omega|(\mathbf{x}_0^+)^{(\ell+1)}(\mathbf{x}_1^-)^{(\ell)} = \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\n_1+\cdots+n_L=N}} \langle\{n_j\}|\omega^{\sum_j jn_j}\bar{K}_{\ell N}(\{n_j\}),$$

$$(\mathbf{x}_1^-)^{(\ell)}(\mathbf{x}_0^+)^{(\ell+1)}|\bar{\Omega}\rangle = -\sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\n_1+\cdots+n_L=N}} K_{\ell N}(\{n_j\})|\{N-1-n_j\}\rangle,$$

$$\langle\bar{\Omega}|(\mathbf{x}_1^-)^{(\ell+1)}(\mathbf{x}_0^+)^{(\ell)} = -\sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\n_1+\cdots+n_L=N}} \langle\{N-1-n_j\}|\bar{K}_{\ell N}(\{n_j\}), \tag{59}$$

where

$$K_m(\{n_j\}) \equiv \sum_{\substack{\{0\leqslant n'_j\leqslant N-1\}\\n'_1+\cdots+n'_L=m}} \prod_{j=1}^{L}\begin{bmatrix}n_j+n'_j\\n'_j\end{bmatrix}\omega^{n'_j N_j}, \qquad N_j \equiv \sum_{\ell=1}^{j-1}n_\ell, \tag{60}$$

$$\bar{K}_m(\{n_j\}) \equiv \sum_{\substack{\{0\leqslant n'_j\leqslant N-1\}\\n'_1+\cdots+n'_L=m}} \prod_{j=1}^{L}\begin{bmatrix}n_j+n'_j\\n'_j\end{bmatrix}\omega^{n'_j \bar{N}_j}, \qquad \bar{N}_j \equiv \sum_{\ell=j+1}^{L}n_\ell, \tag{61}$$

for integers $m \leqslant (N-1)L$. Also we have $\sum_j n'_j N_j = \sum_j n_j \bar{N}'_j$, $\sum_j n'_j \bar{N}_j = \sum_j n_j N'_j$. The generating function of $K_m(\{n_j\})$ for $n_1+\cdots+n_L=kN$ is

$$g(\{n_j\},t) \equiv \sum_{m=0}^{(N-1)L-kN} K_m(\{n_j\})t^m = \frac{1}{(1-t^N)^k}\prod_{j=1}^{L}\frac{1-t^N}{1-t\omega^{N_j}}, \tag{62}$$

which is obtained by inserting $t^{n'_j}$ in each of the L sums of $K_m(\{n_j\})$ and summing over m to remove the constraint; after using (58) and other ω-binomial identities we arrive at (62). Similarly, we can define $\bar{g}(\{n_j\},t)$ with $K_m(\{n_j\})$ replaced by $\bar{K}_m(\{n_j\})$ and N_j by $\bar{N}_j$. As seen from (60) and (61), here we only need to consider the case with $k=1$.

We define the following polynomials for $n_1+\cdots+n_L=N$.

$$G_Q(\{n_j\},z) \equiv \sum_{\ell=0}^{m_Q-1} K_{Q+\ell N}(\{n_i\})z^\ell = \frac{t^{-Q}}{N(1-t^N)}\sum_{a=0}^{N-1}\omega^{-Qa}\prod_{j=1}^{L}\frac{1-t^N}{1-t\omega^{a+N_j}}, \tag{63}$$

and

$$\bar{G}_Q(\{n_j\},z) \equiv \sum_{\ell=0}^{m_Q-1} \bar{K}_{Q+\ell N}(\{n_j\})z^\ell = \frac{t^{-Q}}{N(1-t^N)}\sum_{a=0}^{N-1}\omega^{-Qa}\prod_{j=1}^{L}\frac{1-t^N}{1-t\omega^{a+\bar{N}_j}}, \tag{64}$$

where $z=t^N$. The last equalities in (63) and (64) are proved noting

$$G_Q(\{n_j\},z) = N^{-1}\sum_{a=0}^{N-1}(t\omega^a)^{-Q}g(\{n_j\},t\omega^a), \tag{65}$$

and a similar equation for $\bar{G}_Q(\{n_j\},z)$. For $Q=0$, we drop the subscripts in $G_0(\{n_j\},z)$ and $\bar{G}_0(\{n_j\},z)$. It is easy to see that $\bar{G}(\{n_j\},z)$ is the complex conjugate of $G(\{n_j\},z)$ when t is real.

Substituting (59) into (53) and (54), and using (63) and (64), we obtain

$$\langle\Omega|\mathbf{E}_m^- = -\beta_{m,0}\sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\n_1+\cdots+n_L=N}} \langle\{n_j\}|\,\omega^{\sum_j jn_j}\bar{G}(\{n_j\},z_m), \tag{66}$$

$$\langle\bar{\Omega}|\mathbf{E}_m^+ = -\beta_{m,0}z_m \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\ n_1+\cdots+n_L=N}} \langle\{N-1-n_j\}|\,\bar{G}(\{n_j\},z_m), \tag{67}$$

$$\mathbf{E}_k^+|\Omega\rangle = \beta_{k,0}z_k \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\ n_1+\cdots+n_L=N}} \omega^{-\sum_j jn_j} G(\{n_j\},z_k)|\{n_j\}\rangle, \tag{68}$$

$$\mathbf{E}_k^-|\bar{\Omega}\rangle = \beta_{k,0} \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\ n_1+\cdots+n_L=N}} G(\{n_j\},z_k)|\{N-1-n_j\}\rangle. \tag{69}$$

Since

$$\langle\Omega|\mathbf{E}_m^-\mathbf{E}_k^+|\Omega\rangle = \delta_{m,k}, \tag{70}$$

we conclude that

$$\beta_{m,0}\beta_{k,0}z_k \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\ n_1+\cdots+n_L=N}} \bar{G}(\{n_j\},z_m)G(\{n_j\},z_k) = -\delta_{mk}. \tag{71}$$

Next, we introduce the polynomials

$$\mathrm{h}_k(z) \equiv \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\ n_1+\cdots+n_L=N}} \bar{G}(\{n_j\},z_k)G(\{n_j\},z). \tag{72}$$

Since the degree of polynomial $G(\{n_j\},z)$ defined in (63) is $r-1$, the degree of $\mathrm{h}_k(z)$ is also $r-1$. From (71) we know its $r-1$ roots; thus $\mathrm{h}_k(z)$ is proportional to $f_k(z)$ defined in (10). In fact, we have

$$\mathrm{h}_k(z) = -f_k(z)/z_k\beta_{k,0}^2 = \beta_{k,0}^{-1}\prod_{\ell\neq k}(z-z_\ell), \tag{73}$$

where we have used $f_k(0) = \beta_{k,0}$ and $\prod_{\ell=1}^r(-z_\ell) = 1$. Similarly, we have

$$\bar{\mathrm{h}}_k(z) \equiv \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\ n_1+\cdots+n_L=N}} \bar{G}(\{n_j\},z)G(\{n_j\},z_k) = \beta_{k,0}^{-1}\prod_{\ell\neq k}(z-z_\ell). \tag{74}$$

We now know enough to calculate the matrix elements we need. Using (66) we find

$$\langle\Omega|\mathbf{E}_m^-\mathcal{T}_0(x_q,y_q)|\Omega\rangle = -\beta_{m,0} \sum_{\substack{\{0\leqslant n_m\leqslant N-1\}\\ n_1+\cdots+n_L=N}} \omega^{\sum_j jn_j}\bar{G}(\{n_j\},z_m)\langle\{n_j\}|\mathcal{T}_0(x_q,y_q)|\Omega\rangle. \tag{75}$$

Comparing (37) with (63) for $Q=0$ and using $\sum_j N_j = \sum_j(L-j)n_j$, we find

$$\langle\{n_i\}|\mathcal{T}_0(x_q,y_q)|\Omega\rangle = N^{-\frac{1}{2}L}\omega^{-\sum_j jn_j}\sum_{a=0}^{N-1}\prod_{j=1}^{L}\frac{y_p^N-x_q^N}{y_p-x_q\omega^{a+N_j}}$$

$$= N^{1-\frac{1}{2}L}\omega^{-\sum_j jn_j}y_p^{rN}\left(1-x_q^N/y_p^N\right)G(\{n_i\},(x_q/y_p)^N). \tag{76}$$

Using (72), (73) and (76), (75) becomes

$$\langle\Omega|\mathbf{E}_m^-\mathcal{T}_0(x_q,y_q)|\Omega\rangle = -\beta_{m,0}y_p^{rN}\left(1-x_q^N/y_p^N\right)N^{1-\frac{1}{2}L}\,\mathrm{h}_m\left(x_q^N/y_p^N\right)$$

$$= -y_p^{rN}\left(1-x_q^N/y_p^N\right)N^{1-\frac{1}{2}L}\prod_{\ell\neq m}\left(x_q^N/y_p^N-z_\ell\right). \tag{77}$$

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From (43), we find

$$\langle\Omega|\mathcal{T}_0(x_q, y_q)|\Omega\rangle = \prod_{j=1}^{r}(X_j + Y_j), \qquad \langle\Omega|\mathbf{E}_m^{-}\mathcal{T}_0(x_q, y_q)|\Omega\rangle = Z_m \prod_{j\neq m}(X_j + Y_j). \tag{78}$$

Now equating the ratios given by (78), (39) with (I.44), and (77) we obtain

$$\frac{X_m + Y_m}{Z_m} = \frac{x_q^N - y_p^N z_m}{x_q^N - y_p^N}. \tag{79}$$

Similarly, we use first (67) and (38), followed by (63), (72) and (73), to find

$$\langle\bar{\Omega}|\mathbf{E}_m^{+}\mathcal{T}_0(x_q, y_q)|\bar{\Omega}\rangle = -\beta_{m,0}z_m(\mu_p x_p/\mu_q)^{rN} N^{-\frac{1}{2}L}$$

$$\times \sum_{\substack{\{0\leqslant n_j\leqslant N-1\}\\n_1+\cdots+n_L=N}} \bar{G}(\{n_j\}, z_m) \sum_{a=0}^{N-1}\prod_{j=1}^{L} \frac{1 - (y_q/x_p)^N}{1 - \omega^{-a-j-N_j'}y_q/x_p}$$

$$= -z_m(\mu_p x_p/\mu_q)^{rN}\left(1 - y_q^N/x_p^N\right)N^{1-\frac{1}{2}L}\prod_{\ell\neq m}\left(y_q^N/x_p^N - z_\ell\right), \tag{80}$$

where $N_j' \equiv \sum_{\ell=1}^{j-1}(N - 1 - n_\ell) = N(j - 1) + 1 - j - N_j$. Using

$$\langle\bar{\Omega}| = (-1)^r\langle\Omega|\left(\mathbf{x}_0^+\right)^{(r)} = \langle\Omega|\prod_{j=1}^{r}\mathbf{E}_j^{-}, \qquad |\bar{\Omega}\rangle = (\mathbf{x}_0^-)^{(r)}|\Omega\rangle = \prod_{j=1}^{r}\mathbf{E}_j^{+}|\Omega\rangle, \tag{81}$$

cf (I.35) and (17), we obtain from (43)

$$\langle\bar{\Omega}|\mathcal{T}_0(x_q, y_q)|\bar{\Omega}\rangle = \prod_{j=1}^{r}(X_j - Y_j), \qquad \langle\bar{\Omega}|\mathbf{E}_m^{+}\mathcal{T}_0(x_q, y_q)|\bar{\Omega}\rangle = Z_m \prod_{j\neq m}(X_j - Y_j). \tag{82}$$

Hence, combining this with (41), (I.44) and (80), one finds

$$\frac{X_m - Y_m}{Z_m} = \frac{x_p^N - y_q^N z_m^{-1}}{x_p^N - y_q^N}. \tag{83}$$

This shows that the rotations $\mathcal{R}$ and $\mathcal{S}$ in (26) can be written as direct products of r two by two matrices $\mathcal{R}_j$ and $\mathcal{S}_j$,

$$\mathcal{S} = \prod_{j=1}^{r}\mathcal{S}_j, \qquad \mathcal{R} = \prod_{j=1}^{r}\mathcal{R}_j, \qquad \mathcal{S}_j(A_j - \mathbf{H}_j B_j)\mathcal{R}_j^{-1} = X_j - \mathbf{H}_j Y_j + \left(\mathbf{E}_j^{+} + \mathbf{E}_j^{-}\right)Z_j. \tag{84}$$

Since $\mathcal{R}$ and $\mathcal{S}$ must be independent of λ_q, the left-hand side of the last equation above is linear in λ_q^{-1}, as seen from (9), and so must its right-hand side be. Thus, the ratios given in (79), (83) and (I.2) yield the following:

$$X_j + Y_j = \epsilon_j k\left(y_p^N z_j - x_q^N\right) = \epsilon_j\left[\left(1 - k'\lambda_p\right)z_j - \left(1 - k'\lambda_q^{-1}\right)\right],$$

$$Z_j = \epsilon_j k\left(y_p^N - x_q^N\right) = \epsilon_j\lambda_p\lambda_q^{-1}k\left(y_q^N - x_p^N\right) = \epsilon_j k'\left(\lambda_q^{-1} - \lambda_p\right), \tag{85}$$

$$X_j - Y_j = \epsilon_j\lambda_p\lambda_q^{-1}k\left(y_q^N z_j^{-1} - x_p^N\right) = \epsilon_j\left[-k'\lambda_p z_j^{-1} + \lambda_q^{-1}\left(k' + \lambda_p z_j^{-1} - \lambda_p\right)\right].$$

We have inserted factors $\lambda_p\lambda_q^{-1}$ in both the numerator and the denominator of the second ratio in (83) so that they become linear in λ_q^{-1}. Since $\det(\mathcal{R}_j) = 1$ and $\det(\mathcal{S}_j) = 1$, the determinant is invariant under the transformation in (84), namely

$$X_j^2 - Y_j^2 - Z_j^2 = A_j^2 - B_j^2. \tag{86}$$

Using (9), (6) and (5) in this order, we find

$$A_j^2 - B_j^2 = \rho^2 k^2 \left(t_p^N z_j - t_q^N\right) / (k' \lambda_q), \tag{87}$$

whereas from equations on the left of (85) and (I.5), we obtain

$$X_j^2 - Y_j^2 - Z_j^2 = \epsilon_j^2 k^2 \lambda_q^{-1} \lambda_p \left(z_j^{-1} - 1\right) \left(t_p^N z_j - t_q^N\right). \tag{88}$$

Consequently we find the multiplicative constants

$$\epsilon_j^2 = \bar{\epsilon}_j^2 \rho^2 = \rho^2 / \left[k' \left(z_j^{-1} - 1\right) \lambda_p\right], \qquad \rho^2 = N^{1/r} k' / k^2. \tag{89}$$

Since the right-hand side of (84) is explicitly given in (85) and the λ_q dependence in the left-hand side comes only from A_j and B_j in (9), we conclude that both sides are linear in λ_q^{-1}. Therefore, we can get two equations by equating the coefficients of the constant and linear terms of (84) separately, i.e.

$$\mathcal{S}_j (\cosh \theta_j \mathbf{1} - \sinh \theta_j \mathbf{H}_j) \mathcal{R}_j^{-1} = \mathbf{M}, \qquad \mathcal{S}_j (\cosh \theta_j \mathbf{1} + \sinh \theta_j \mathbf{H}_j) \mathcal{R}_j^{-1} = -\mathbf{N}, \tag{90}$$

where the matrices $\mathbf{M}$ and $\mathbf{N}$ depend on j and have elements

$$m_{11} = -\bar{\epsilon}_j k' \lambda_p / z_j, \qquad m_{12} = m_{21} = -\bar{\epsilon}_j k' \lambda_p, \qquad m_{22} = \bar{\epsilon}_j (z_j - 1 - k' z_j \lambda_p),$$
$$n_{11} = \bar{\epsilon}_j \left(z_j^{-1} \lambda_p - \lambda_p + k'\right), \qquad n_{12} = n_{21} = n_{22} = \bar{\epsilon}_j k', \qquad k' \left(z_j^{-1} - 1\right) \lambda_p \bar{\epsilon}_j^2 = 1. \tag{91}$$

It is straightforward to solve for $\mathcal{R}_j$ and $\mathcal{S}_j$. We shall leave the details to the appendix and present the results here. We find that

$$\mathcal{S}_j = \tfrac{1}{2}(s_{11} + s_{22})\mathbf{1} + \tfrac{1}{2}(s_{11} - s_{22})\mathbf{H}_j + s_{12}\mathbf{E}_j^+ + s_{21}\mathbf{E}_j^-, \tag{92}$$

with

$$s_{22} = \left(\frac{m_{22}\, e^{\theta_j} + n_{22}\, e^{-\theta_j}}{2 \sinh 2\theta_j}\right)^{1/2}, \qquad s_{12} = \frac{m_{12}\, e^{\theta_j} + n_{12}\, e^{-\theta_j}}{m_{22}\, e^{\theta_j} + n_{22}\, e^{-\theta_j}} s_{22},$$
$$s_{21} = \frac{e^{-2\theta_j} - k'}{2 s_{12} \sinh 2\theta_j}, \qquad s_{11} = \frac{e^{2\theta_j} - k'}{2 s_{22} \sinh 2\theta_j}, \tag{93}$$

where the θ_j are defined in (6), while $\mathcal{R}_j$ is the transpose of the inverse of $\mathcal{S}_j$, that is

$$r_{22} = s_{11}, \qquad r_{21} = -s_{12}, \qquad r_{12} = -s_{21}, \qquad r_{11} = s_{22}. \tag{94}$$

We now show that $\mathcal{R}$ and $\mathcal{S}$ in equation (27) for the other two transfer matrices are identical. To calculate the matrix elements of $\hat{\mathcal{T}}_Q(y_q, x_q)$ we use (2) and (I.4), while using an equation like (34), but with the replacements $p \leftrightarrow p'$, $q \leftrightarrow q'$ (or $x_r \leftrightarrow y_r$ and $\mu_r \to \mu_r^{-1}$ with $r = p, q$). Similar to (33) and (35), we may then write

$$\langle \{n_i'\} | \hat{\mathcal{T}}_Q(y_q, x_q) | \{n_i\} \rangle = N^{-\frac{1}{2}L} \sum_{a=0}^{N-1} \omega^{-Qa}$$
$$\times \prod_{j=1}^{L} \left[\frac{(x_p^N - y_q^N) \omega^{a - N_j + N_j'}}{x_p - y_q \omega^{a - N_j + N_j'}} \left(\frac{\mu_q}{\mu_p}\right)^{n_j'} \prod_{\ell=1}^{n_j'} \frac{x_q - y_p \omega^{\ell + a - N_j + N_j'}}{x_p - y_q \omega^{\ell + a - N_j + N_j'}} \right]. \tag{95}$$

Here, when $n_i = 0$ or $n_i = N - 1$, the above expressions are related to the functions defined in (63) and (64), i.e.

$$\langle \Omega | \hat{\mathcal{T}}_0(y_q, x_q) | \{n_i\} \rangle = N^{1 - \frac{1}{2}L} \omega^{\sum_{j=1}^{L} j n_j} x_p^{rN} \left(1 - y_q^N / x_p^N\right) \bar{G}(\{n_i\}, (y_q / x_p)^N), \tag{96}$$

$$\langle \bar{\Omega} | \hat{\mathcal{T}}_0(y_q, x_q) | \{N - 1 - n_i\} \rangle = N^{1 - \frac{1}{2}L} (y_p \mu_q / \mu_p)^{rN} \left(1 - x_q^N / y_p^N\right) \bar{G}(\{n_i\}, (x_q / y_p)^N). \tag{97}$$

Combining this with (68), (69) and (74), and also noting that $(1 - z)\bar{G}(\{0\}, z) = P(z)$ for $n_j \equiv 0$, we can derive

$$\frac{\langle\Omega|\hat{\mathcal{T}}_0(y_q, x_q)|\Omega\rangle}{\langle\Omega|\hat{\mathcal{T}}_0(y_q, x_q)\mathbf{E}_m^+|\Omega\rangle} = -\frac{x_p^N - y_q^N z_m^{-1}}{x_p^N - y_q^N}, \qquad \frac{\langle\bar{\Omega}|\hat{\mathcal{T}}_0(y_q, x_q)|\bar{\Omega}\rangle}{\langle\bar{\Omega}|\hat{\mathcal{T}}_0(y_q, x_q)\mathbf{E}_m^-|\bar{\Omega}\rangle} = -\frac{x_q^N - y_p^N z_m}{x_q^N - y_p^N}. \quad (98)$$

Setting $x_q \leftrightarrow y_q$ in (98), we find

$$\frac{\langle\Omega|\hat{\mathcal{T}}_0(x_q, y_q)|\Omega\rangle}{\langle\Omega|\hat{\mathcal{T}}_0(x_q, y_q)\mathbf{E}_m^+|\Omega\rangle} = -\frac{x_p^N - x_q^N z_m^{-1}}{x_p^N - x_q^N}, \qquad \frac{\langle\bar{\Omega}|\hat{\mathcal{T}}_0(x_q, y_q)|\bar{\Omega}\rangle}{\langle\bar{\Omega}|\hat{\mathcal{T}}_0(x_q, y_q)\mathbf{E}_m^-|\bar{\Omega}\rangle} = -\frac{y_q^N - y_p^N z_m}{y_q^N - y_p^N}. \quad (99)$$

Similar to (43) and (84), we find that (27) can be written as

$$\hat{\mathcal{T}}_0(y_q, x_q) = \prod_{j=1}^{r}\left[\bar{X}_j - \mathbf{H}_j\bar{Y}_j + (\mathbf{E}_j^+ + \mathbf{E}_j^-)\bar{Z}_j\right] = \prod_{j=1}^{r}\mathcal{R}_j(\bar{A}_j - \mathbf{H}_j\bar{B}_j)\mathcal{S}_j^{-1}, \quad (100)$$

$$\hat{\mathcal{T}}_0(x_q, y_q) = \prod_{j=1}^{r}\left[X_j' - \mathbf{H}_j Y_j' + (\mathbf{E}_j^+ + \mathbf{E}_j^-)Z_j'\right] = \prod_{j=1}^{r}\mathcal{R}_j(A_j - \mathbf{H}_j B_j)\mathcal{S}_j^{-1}. \quad (101)$$

Since $\bar{A}_j$ and $\bar{B}_j$ are obtained from A_j and B_j in (9) by letting $\lambda_q^{-1} \to \lambda_q$ as follows from (I.2) for $\lambda_q \equiv \mu_q^N$, they are linear in λ_q, so that, similar to (85), (98) yields

$$\bar{X}_j + \bar{Y}_j = \epsilon_j\lambda_p k(x_p^N - y_q^N z_j^{-1}), \quad \bar{Z}_j = -\epsilon_j\lambda_p k(x_p^N - y_q^N) = -\epsilon_j\lambda_q k(x_q^N - y_p^N),$$
$$\bar{X}_j - \bar{Y}_j = \epsilon_j\lambda_q k(x_q^N - y_p^N z_j), \quad (102)$$

while each term in the product of (101) is linear in λ_q^{-1}, so that (99) leads to

$$X_j' + Y_j' = \epsilon_j\lambda_p k(x_p^N - x_q^N z_j^{-1}), \quad Z_j' = -\epsilon_j\lambda_p k(x_p^N - x_q^N) = -\epsilon_j\lambda_q^{-1}k(y_q^N - y_p^N),$$
$$X_j' - Y_j' = \epsilon_j\lambda_q^{-1}k(y_q^N - y_p^N z_j). \quad (103)$$

By equating separately the coefficients of the terms constant and linear in λ_q of the jth factors of the products on both sides of (100), we find

$$\mathcal{R}_j(\cosh\theta_j\mathbf{1} - \sinh\theta_j\mathbf{H}_j)\mathcal{S}_j^{-1} = -\mathbf{N}^{-1}, \qquad \mathcal{R}_j(\cosh\theta_j\mathbf{1} + \sinh\theta_j\mathbf{H}_j)\mathcal{S}_j^{-1} = \mathbf{M}^{-1}, \quad (104)$$

where the elements of the matrices $\mathbf{M}$ and $\mathbf{N}$ were given in (91) and for their inverses we used $\det\mathbf{M} = \det\mathbf{N} = 1$. From (101) we get the same equation (104), as this case differs only by the interchange $q \leftrightarrow q'$, $\lambda_q \leftrightarrow 1/\lambda_q$. Taking the inverses of the two equations in (104) we recover (90). This proves that we can indeed use the identical $\mathcal{R}_j$ and $\mathcal{S}_j$.

To summarize our results, in (43) and (101) we have expressed the transfer matrices of the superintegrable chiral Potts model for the $Q = 0$ ground-state sector in terms of the generators of $\mathfrak{sl}_2$ algebras, with constants X_j, Y_j, Z_j and X_j', Y_j', Z_j' given explicitly in (85) and (103), where $j = 1, \ldots, r$ with $r = (N-1)L/N$ and L a multiple of N. The corresponding two sets of 2^r eigenstates for the transfer matrices are given in (28) and (29) with the rotations $\mathcal{R}$ and $\mathcal{S}$ explicitly given in (84) and (92) to (94).

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Appendix A. $\mathbf{H}_m|\Omega\rangle = -|\Omega\rangle$ and $(\mathbf{E}_m^+)^2|\Omega\rangle = 0$

After applying x_0^+ to (I.42), we find, using (I.37) and $\mathbf{h}_n = x_0^+ x_n^- - x_n^- x_0^+$ in (I.34),

$$n\Lambda_n = \sum_{j=1}^{n} d_j \Lambda_{n-j}, \qquad \text{where} \quad \mathbf{h}_n|\Omega\rangle = d_n|\Omega\rangle. \tag{A.1}$$

Similar to what is done in [7], we multiply both side by z^n and then sum over n from 1 to ∞. Next, noting $\Lambda_n = 0$ for $n > r$, we divide the result by $P(z)$ and obtain

$$z\frac{d}{dz}\ln P(z) = \sum_{n=1}^{\infty} d_n z^n = \sum_{j=1}^{r} \frac{z}{z-z_j} = -\sum_{n=1}^{\infty} z^n \sum_{j=1}^{r} z_j^{-n}. \tag{A.2}$$

This shows $d_n = -\sum_{j=1}^{r} z_j^{-n}$. Using (11) we obtain

$$\sum_{n=0}^{r-1} \beta_{m^*,n} d_{n+k} = -\sum_{j=1}^{r} \sum_{n=0}^{r-1} \beta_{m^*,n} z_{j^*}^{n+k} = -\sum_{j=1}^{r} \delta_{m,j} z_j^{-k} = -z_m^{-k}. \tag{A.3}$$

From (13), we find

$$\mathbf{H}_m|\Omega\rangle = \sum_{n=0}^{r-1} \beta_{m^*,n} d_n|\Omega\rangle = -|\Omega\rangle. \tag{A.4}$$

From (15) we see that $\mathbf{E}_m^\pm$ and $\mathbf{H}_m$ generate a finite-dimensional representation of $\mathfrak{sl}(2)$. We also have $\mathbf{E}_m^-|\Omega\rangle = 0$. Thus this representation is spin-$\frac{1}{2}$, proving $(\mathbf{E}_m^+)^2|\Omega\rangle = 0$.

Appendix B. Results for $\mathbf{E}_j^+ \mathbf{E}_m^+ |\Omega\rangle$

As the identities in (13) hold for the entire eigenspace, whereas the identity in (54) holds only for the ground state, we use (13) with $\ell = 0$ for $\mathbf{E}_j^+$ and (54) for $\mathbf{E}_m^+$ to find

$$\mathbf{E}_j^+ \mathbf{E}_m^+ |\Omega\rangle = \beta_{m,0} \sum_{\ell=1}^{r} z_m^\ell \sum_{\ell'=0}^{r-1} \beta_{j^*,\ell'} \mathbf{x}_{\ell'}^- (\mathbf{x}_0^+)^{(\ell-1)} (\mathbf{x}_1^-)^{(\ell)} |\Omega\rangle. \tag{B.1}$$

As $(\mathbf{x}_m^\pm)^{(n)} \equiv (\mathbf{x}_m^\pm)^n / n!$ for $n \geq 0$ and $(\mathbf{x}_m^\pm)^{(n)} \equiv 0$ for $n < 0$, it is easy to prove by induction that the following relations hold for the entire eigenspace,

$$
\begin{aligned}
&\left[(\mathbf{x}_0^+)^{(j)}, \mathbf{x}_k^- \right] = (\mathbf{x}_0^+)^{(j-1)} \mathbf{h}_k - \mathbf{x}_k^+ (\mathbf{x}_0^+)^{(j-2)}, \\
&\left[\mathbf{x}_k^+, (\mathbf{x}_1^-)^{(j)} \right] = (\mathbf{x}_1^-)^{(j-1)} \mathbf{h}_{k+1} + \mathbf{x}_{k+2}^- (\mathbf{x}_1^-)^{(j-2)}, \\
&\left[\mathbf{h}_k, (\mathbf{x}_0^+)^{(j)} \right] = -2\mathbf{x}_k^+ (\mathbf{x}_0^+)^{(j-1)}, \quad \left[\mathbf{h}_k, (\mathbf{x}_1^-)^{(j)} \right] = 2\mathbf{x}_{k+1}^- (\mathbf{x}_0^-)^{(j-1)},
\end{aligned} \tag{B.2}
$$

which are very similar to those in [7]. Using (B.2) and (A.1) we interchange the order of operations in (B.1), i.e.

$$
\begin{aligned}
\mathbf{x}_{\ell'}^- (\mathbf{x}_0^+)^{(\ell-1)} (\mathbf{x}_1^-)^{(\ell)} |\Omega\rangle = &\{ (\mathbf{x}_0^+)^{(\ell-1)} (\mathbf{x}_1^-)^{(\ell)} \mathbf{x}_{\ell'}^- - (\mathbf{x}_0^+)^{(\ell-2)} [2(\mathbf{x}_1^-)^{(\ell-1)} \mathbf{x}_{\ell'+1}^- \\
&+ (\mathbf{x}_1^-)^{(\ell)} d_{\ell'}] + (\mathbf{x}_0^+)^{(\ell-3)} [(\mathbf{x}_1^-)^{(\ell-2)} \mathbf{x}_{\ell'+2}^- + (\mathbf{x}_1^-)^{(\ell-1)} d_{\ell'+1}] \} |\Omega\rangle.
\end{aligned} \tag{B.3}
$$

Substituting (B.3) into (B.1), and using (13) and (A.3), we get

$$\mathbf{E}_j^+\mathbf{E}_m^+|\Omega\rangle = \beta_{m,0}\Bigg\{\sum_{\ell=1}^{r} z_m^\ell (\mathbf{x}_0^+)^{(\ell-1)}(\mathbf{x}_1^-)^{(\ell)}\mathbf{E}_j^+|\Omega\rangle$$

$$-\sum_{\ell=2}^{r} z_m^\ell (\mathbf{x}_0^+)^{(\ell-2)}\big[(\mathbf{x}_1^-)^{(\ell-1)}2z_j^{-1}\mathbf{E}_j^+ - (\mathbf{x}_1^-)^{(\ell)}\big]|\Omega\rangle$$

$$+\sum_{\ell=3}^{r} z_m^\ell (\mathbf{x}_0^+)^{(\ell-3)}\big[(\mathbf{x}_1^-)^{(\ell-2)}z_j^{-2}\mathbf{E}_j^+ - (\mathbf{x}_1^-)^{(\ell-1)}z_j^{-1}\big]|\Omega\rangle\Bigg\}$$

$$= \beta_{m,0}\Bigg\{(1 - 2z_m/z_j + z_m^2/z_j^2)\sum_{\ell=1}^{r-1} z_m^\ell (\mathbf{x}_0^+)^{(\ell-1)}(\mathbf{x}_1^-)^{(\ell)}\mathbf{E}_j^+|\Omega\rangle$$

$$+ (1 - z_m/z_j)\sum_{\ell=2}^{r} z_m^\ell (\mathbf{x}_0^+)^{(\ell-2)}(\mathbf{x}_1^-)^{(\ell)}|\Omega\rangle$$

$$+ z_m^r (\mathbf{x}_0^+)^{(r-1)}(\mathbf{x}_1^-)^{(r)}\mathbf{E}_j^+|\Omega\rangle$$

$$- z_m^{r+1}(\mathbf{x}_0^+)^{(r-2)}\big[z_j^{-2}(\mathbf{x}_1^-)^{(r-1)}\mathbf{E}_j^+ - (\mathbf{x}_1^-)^{(r)}z_j^{-1}\big]|\Omega\rangle\Bigg\}. \tag{B.4}$$

To arrive at the second equality, we changed the summations in the first equality replacing $\ell - 1 \to \ell$ both in the first part of the second sum and in the second part of the third sum, and $\ell - 2 \to \ell$ in the first part of the third sum. If we can use the identities

$$(\mathbf{x}_1^-)^{(r)}\mathbf{E}_j^+|\Omega\rangle = 0, \qquad (\mathbf{x}_1^-)^{(r-1)}\mathbf{E}_j^+|\Omega\rangle = z_j(\mathbf{x}_1^-)^{(r)}|\Omega\rangle, \tag{B.5}$$

we may simplify (B.4) to

$$\mathbf{E}_j^+\mathbf{E}_m^+|\Omega\rangle = \beta_{m,0}\Bigg\{(1 - z_m/z_j)^2 \sum_{\ell=1}^{r-1} z_m^\ell (\mathbf{x}_0^+)^{(\ell-1)}(\mathbf{x}_1^-)^{(\ell)}\mathbf{E}_j^+|\Omega\rangle$$

$$+ (1 - z_m/z_j)\sum_{\ell=2}^{r} z_m^\ell (\mathbf{x}_0^+)^{(\ell-2)}(\mathbf{x}_1^-)^{(\ell)}|\Omega\rangle\Bigg\}. \tag{B.6}$$

This is clearly consistent with $(\mathbf{E}_m^+)^2|\Omega\rangle = 0$.

From (68), (I.35) and (55), it is easy to see that the first identity in (B.5) holds. To prove the second identity, we use (56) with $n_j \equiv N - 1$ to find

$$(\mathbf{x}_1^-)^{(r)}|\Omega\rangle = \omega^{\frac{1}{2}L(L+1)}|\bar\Omega\rangle. \tag{B.7}$$

Next use (68), (I.35) with $\nu_m = N - 1 - n_m$, (55), and finally (58) with $n = 0$, to find

$$(\mathbf{x}_1^-)^{(r-1)}\mathbf{E}_j^+|\Omega\rangle = \omega^{\frac{1}{2}L(L+1)}|\bar\Omega\rangle(-z_j\beta_{j,0})\sum_{\substack{\{0\leqslant n_m \leqslant N-1\}\\ n_1+\cdots+n_L=N}} \omega^{-\sum_m mn_m}G(\{n_m\}, z_j). \tag{B.8}$$

Multiplying both sides of (74) by z^{-r+1} and then taking the limit $z \equiv t^N \to \infty$, using (64) to obtain the limit $t^{-1} \to 0$ of $t^{-Nr+N}\bar{G}$, where $\sum_m \bar{N}_m = \sum_m (m-1)n_m$, we find

$$\lim_{z\to\infty} z^{-r+1}\bar{\mathsf{h}}_k(z) = -\sum_{\substack{\{0\leqslant n_j \leqslant N-1\}\\ n_1+\cdots+n_L=N}} \omega^{-\sum_m mn_m}G(\{n_j\}, z_k) = -\beta_{k,0}^{-1}. \tag{B.9}$$

Substituting this equation into (B.8), we show that the second identity in (B.5) holds.

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Appendix C. The rotations $\mathbf{R}_j$ and $\mathbf{S}_j$

We rewrite (90) as

$$\mathcal{S}_j(\cosh\theta_j\mathbf{1} - \sinh\theta_j\mathbf{H}_j) = \mathbf{M}\mathcal{R}_j, \qquad \mathcal{S}_j(\cosh\theta_j\mathbf{1} + \sinh\theta_j\mathbf{H}_j) = -\mathbf{N}\mathcal{R}_j, \tag{C.1}$$

where $\det(\mathbf{M}) = \det(\mathbf{N}) = 1$. We multiply the first equation with $\mathbf{M}^{-1}$ and the second equation with $-\mathbf{N}^{-1}$, so that the right-hand sides of both equations become $\mathcal{R}_j$ and the left-hand sides must be equal. This way we obtain a matrix equation for $\mathcal{S}_j$. More explicitly its elements s_{ij} can be shown to satisfy the following equations:

$$\begin{aligned}
s_{21}(m_{11}\,\mathrm{e}^{-\theta_j} + n_{11}\,\mathrm{e}^{\theta_j}) &= s_{11}(m_{21}\,\mathrm{e}^{-\theta_j} + n_{21}\,\mathrm{e}^{\theta_j}), \\
s_{21}(m_{12}\,\mathrm{e}^{-\theta_j} + n_{12}\,\mathrm{e}^{\theta_j}) &= s_{11}(m_{22}\,\mathrm{e}^{-\theta_j} + n_{22}\,\mathrm{e}^{\theta_j}), \\
s_{22}(m_{11}\,\mathrm{e}^{\theta_j} + n_{11}\,\mathrm{e}^{-\theta_j}) &= s_{12}(m_{21}\,\mathrm{e}^{\theta_j} + n_{21}\,\mathrm{e}^{-\theta_j}), \\
s_{22}(m_{12}\,\mathrm{e}^{\theta_j} + n_{12}\,\mathrm{e}^{-\theta_j}) &= s_{12}(m_{22}\,\mathrm{e}^{\theta_j} + n_{22}\,\mathrm{e}^{-\theta_j}).
\end{aligned} \tag{C.2}$$

These equations are consistent if and only if

$$(m_{11}\,\mathrm{e}^{\mp\theta_j} + n_{11}\,\mathrm{e}^{\pm\theta_j})(m_{22}\,\mathrm{e}^{\mp\theta_j} + n_{22}\,\mathrm{e}^{\pm\theta_j}) = (m_{21}\,\mathrm{e}^{\mp\theta_j} + n_{21}\,\mathrm{e}^{\pm\theta_j})(m_{12}\,\mathrm{e}^{\mp\theta_j} + n_{12}\,\mathrm{e}^{\pm\theta_j}). \tag{C.3}$$

We can simplify (C.3) using $m_{12} = m_{21}$ and $n_{12} = n_{21}$ found in (91) and again using $\det(\mathbf{M}) = \det(\mathbf{N}) = 1$. We find that the two conditions become

$$2\cosh 2\theta_j = 2m_{12}n_{12} - m_{11}n_{22} - m_{22}n_{11}. \tag{C.4}$$

Next, using (5) and (6), we can show

$$\lambda_p + \lambda_p^{-1} = 2z_j^{-1}\cosh 2\theta_j - (k' + k'^{-1})(z_j^{-1} - 1). \tag{C.5}$$

With the help of (91) and (C.5) we can then prove that (C.4) holds.

Moreover, from $\det\mathcal{S}_j = 1$ we calculate $s_{11}s_{22}$ eliminating first $s_{12}s_{12}$ using (C.2) and (91), and then substituting z_j from (C.5). This way we find

$$s_{11}s_{22} = \frac{\mathrm{e}^{2\theta_j} - k'}{2\sinh 2\theta_j}, \qquad s_{12}s_{21} = \frac{\mathrm{e}^{-2\theta_j} - k'}{2\sinh 2\theta_j}. \tag{C.6}$$

Following nearly identical steps, we obtain from (C.1) consistent equations similar to (C.2) and (C.6) for the matrix elements r_{ij} of $\mathcal{R}_j$. From these results we conclude

$$\frac{r_{22}}{r_{12}} = -\frac{s_{11}}{s_{21}}, \qquad \frac{r_{11}}{r_{21}} = -\frac{s_{22}}{s_{12}}, \qquad \begin{aligned} r_{22}r_{11} &= s_{11}s_{22}, \\ r_{12}r_{21} &= s_{12}s_{21}. \end{aligned} \tag{C.7}$$

Only six of the equations in (C.2), (C.6) and (C.7) are independent. At this point we may choose s_{22} and r_{22} as free parameters, but then the other six elements are uniquely determined. However, there is one more relation. From direct multiplication of the matrices in (90), we find

$$m_{11} = \mathrm{e}^{-\theta_j}s_{11}r_{22} - \mathrm{e}^{\theta_j}s_{12}r_{21}, \qquad n_{11} = -\mathrm{e}^{\theta_j}s_{11}r_{22} + \mathrm{e}^{-\theta_j}s_{12}r_{21}. \tag{C.8}$$

Eliminating $s_{12}r_{21}$ from these (and similarly starting with m_{22} and n_{22}), we find

$$s_{11}r_{22} = -\frac{m_{11}\,\mathrm{e}^{-\theta_j} + n_{11}\,\mathrm{e}^{\theta_j}}{2\sinh 2\theta_j}, \qquad s_{22}r_{11} = \frac{m_{22}\,\mathrm{e}^{\theta_j} + n_{22}\,\mathrm{e}^{-\theta_j}}{2\sinh 2\theta_j}. \tag{C.9}$$

Let us define

$$T_{ik} \equiv m_{ik}\,\mathrm{e}^{-\theta_j} + n_{ik}\,\mathrm{e}^{\theta_j} = T_{ki}, \qquad T_{ik}^* \equiv m_{ik}\,\mathrm{e}^{\theta_j} + n_{ik}\,\mathrm{e}^{-\theta_j} = T_{ki}^*. \tag{C.10}$$

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Then it can be easily verified, substituting (91) and eliminating z_j from (C.5), that

$$
\begin{aligned}
T_{12}T_{12}^* &= -(\mathrm{e}^{-2\theta_j} - k')(\mathrm{e}^{2\theta_j} - k'), \\
T_{11}T_{22}^* &= -(\mathrm{e}^{2\theta_j} - k')^2, \qquad T_{22}T_{11}^* = -(\mathrm{e}^{-2\theta_j} - k')^2.
\end{aligned}
\tag{C.11}
$$

Using (C.6) and (C.11), we may rewrite (C.9) as

$$
\frac{r_{22}}{s_{22}} = -\frac{T_{11}}{\mathrm{e}^{2\theta_j} - k'} = \frac{\mathrm{e}^{2\theta_j} - k'}{T_{22}^*}.
\tag{C.12}
$$

This shows that there is only one free parameter s_{22}, and we choose it so that $r_{11} = s_{22}$. This choice yields the solution given in (93) and (94).

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